Introduction to Quantum Computing
Lecture 17 & 18: MBQC II & III

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Towards a Universal Resource

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2. How to break a large graph state to smaller:
   The role of the Z-measurement

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   - Continue with MBQC pattern as before

I.e. find U such that $U|+\rangle = |\psi\rangle$, then implement U in MBQC with trivial input and continue

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Need to take into account (random) corrections (see later)

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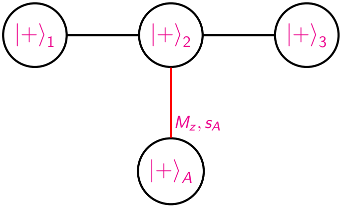
- Can take into account by flipping the value of s_j
- Perform **all** Z measurements first (reducing the resource to the graph needed), and keep track of outcome corrections for the remaining pattern

Example:



Reducing MBQC pattern

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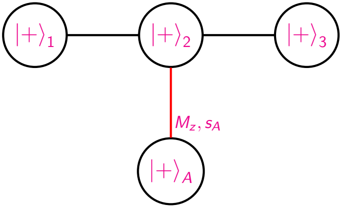

Reducing MBQC pattern

- Consider first the Z measurement at qubit A
- Need to consider first operations that involve qubit A (before measuring):

$$\wedge Z_{A2} |+\rangle_A |+\rangle_2 = |0\rangle_A |+\rangle_2 + |1\rangle_A |-\rangle_2$$

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$$|s_A\rangle \, Z^{s_A} \, |+\rangle$$

since $s_A = 1$ leads to $|-\rangle_2 = Z\,|+\rangle_2$.

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Summary:

- Measuring qubit i in $|\pm_\theta\rangle$ results to a gate $J(-\theta)$ *and* a correction X^{s_i} on the "next" qubit. **Correction affects the new measurement angle** (to cancel it).
- Measuring qubit i in $\{|0\rangle, |1\rangle\}$results to no gate being applied. Only a correction Z^{s_i} applied to *all* neighbours. **Correction changes the labelling of the outcome, but not the measurement**.

Measurement Pattern for Generic Computation

- Corrections appear when $s_i \neq 0$
- Let ϕ_i be the measurement angles that implement the desired unitary if **all** measurements give the result zero $s_i = 0 \ \forall \ i$
- The set $\{\phi_i\}_i$ determine the computation performed, and we call them **default measurement angles**.
- We define $\phi'_i(\phi_i, s_j \ | j \in \ \{\text{past of } i\})$ to be the **corrected measurement angles** (see later for expression)
- Default angles determine computation, corrected angles are the one used for measurements

Graph States as Stabiliser States

Graph state $|G\rangle$ is defined as:

$$|G\rangle = \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V}$$

An operator A stabilises a state if

$$A|\psi\rangle = |\psi\rangle$$

The state $|G\rangle$ is a **stabiliser state** with generators:

$$\boxed{K_i := X_i \left(\prod_{j \in N_G(i)} Z_j \right)}$$

For each vertex $i \in V$ there is a stabiliser that has X at that vertex and Z to all its neighbours $N_G(i)$ in the graph.

Operators K_i stabilise $|G\rangle$:

$$\begin{aligned} K_i\,|G\rangle &= X_i\left(\prod_{j\in N_G(i)} Z_j\right)\prod_{(a,b)\in E}\wedge Z^{(a,b)}\,|+\rangle^{\otimes V} \\ &= X_i\prod_{(a,b)\in E}\wedge Z^{(a,b)}\,|+\rangle^{\otimes V\setminus N_G(i)}\,|-\rangle^{\otimes j\in N_G(i)} \end{aligned}$$

where we used that Z commutes with $\wedge Z$ and that $Z\,|+\rangle = |-\rangle$.

Graph States as Stabiliser States

We know that $X_i \wedge Z^{(i,j)} = \wedge Z^{(i,j)} X_i Z_j$ (see previous lecture) and we get:

$$K_i |G\rangle \quad = \quad \prod_{(a,b)\in E} \wedge Z^{(a,b)} \left(X_i \prod_{j \in N_G(i)} Z_j \right) |+\rangle^{\otimes V \setminus N_G(i)} |-\rangle^{\otimes j \in N_G(i)}$$

since X_i acts as above if i belongs to that edge while it commutes with all the other $\wedge Z$ that do not involve qubit i. However this changes back the states since $Z|-\rangle = |+\rangle$, and $X|+\rangle = |+\rangle$ results to

$$K_i |G\rangle \quad = \quad \prod_{(a,b)\in E} \wedge Z^{(a,b)} |+\rangle^{\otimes V}, \ \forall\, i \in V = |G\rangle$$

- It can be shown that this set of generators uniquely determines the graph state $|G\rangle$.

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Does a consistent order (for measurements and corrections) exist?

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Definition: An entanglement graph (G, I, O) has flow if there exists a map $f : O^c \to I^c$ and a partial order $\preceq$ over qubits

1. $x \sim f(x)$: (x and $f(x)$ are neighbours in the graph)
2. $x \preceq f(x)$: ($f(x)$ is to the future of x with respect to the partial order)
3. for all $y \sim f(x)$, we have $x \preceq y$: (any other neighbours of $f(x)$ are all to the future of x)

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- **Example**: 2-dim lattice. $f(x) \Rightarrow$ same row, next column

How to apply an operator by acting on different qubits

Given a graph state $|G\rangle$, we can apply X, Z operators at qubit i by acting on qubits other than i.

We will use: $K_i |G\rangle = |G\rangle$

1. To apply X_i:

$$X_i |G\rangle = X_i K_i |G\rangle = \prod_{j \in N_G(i)} Z_j |G\rangle$$

   where $N_G(i)$ are the neighbours of i in the graph

2. To apply Z_i:

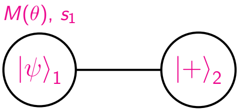
$$\boxed{Z_i |G\rangle = Z_i K_{f(i)} |G\rangle = X_{f(i)} \prod_{j \in N_G(f(i)) \setminus i} Z_j |G\rangle}$$

   where $f(i)$ is the flow of i and $N_G(f(i)) \setminus i$ are all the neighbours of $f(i)$ in the graph apart from i

How to apply an operator by acting on different qubits

Can cancel a correction **after** measuring that qubit, provided **all** qubits j involved are still not measured!

- This **cannot** be done for the X operator (qubits both to the past and future according to the definition of flow).
  Need to adapt the angles
- This **can** be done for the Z operator because of properties of the flow! (conditions 2 and 3)



Measuring at θ qubit 1

Recall (previous lecture) the state before measurement is:

$$
\begin{aligned}
|\psi'\rangle_{12} &= \wedge Z_{12}\left(|\psi\rangle_1 \otimes |+\rangle_2\right) = a\,|0+\rangle_{12} + b\,|1-\rangle_{12} \\
&= |+_\theta\rangle_1\, X^{s_1} J(-\theta)_2\, |\psi\rangle_2 + |-_\theta\rangle_1\, X_2^{s_1} J(-\theta)_2\, |\psi\rangle_2
\end{aligned}
$$

Rules for adapting measurement angles

If we could have started with $Z_1^{s_1}\left|\psi\right\rangle_1$ state instead of $\left|\psi\right\rangle_1$:

$$\begin{aligned}\left|\psi'\right\rangle_{12} &= \left|+_\theta\right\rangle_1 X^{s_1} J(-\theta)_2 Z_2^{s_1}\left|\psi\right\rangle_2 + \left|-_\theta\right\rangle_1 X_2^{s_1} J(-\theta)_2 Z_2^{s_1}\left|\psi\right\rangle_2 \\ &= \left|+_\theta\right\rangle_1 J(-\theta)_2\left|\psi\right\rangle_2 + \left|-_\theta\right\rangle_1 J(-\theta)_2\left|\psi\right\rangle_2\end{aligned}$$

using that $J(-\theta)Z^{s_1} = X^{s_1}J(-\theta)$ and $X^{s_1}X^{s_1} = \mathbf{I}$. Now, there is **no random correction** and any outcome of the measurement leads to the desired gate.

- Getting the "wrong" outcome $s_i = 1$ is as if a Z-correction on the initial state was applied, and could cancel it by applying another Z on that qubit.
- However, to do this we **need to know** s_1 which is the outcome of measuring qubit 1, and this (clearly) happens **after** the preparation of qubit 1.

Rules for adapting measurement angles

- Using stabilisers, we proved that we can apply Z by acting on other qubits that are not yet measured!
- Instead of acting on the non-measured qubits, we can modify the measurement angles:

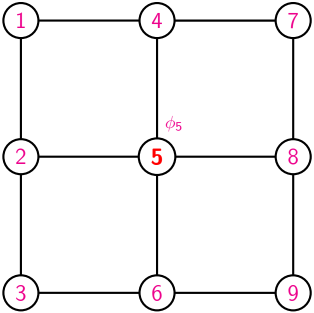
$$M_i^{\phi_i} X = M_i^{-\phi_i} \; ; \; M_i^{\phi_i} Z = M_i^{\phi_i + \pi}$$

since $X\left|+_{\phi_i}\right\rangle = \left|+_{(-\phi_i)}\right\rangle$ and $Z\left|+_{\phi_i}\right\rangle = \left|-_{\phi_i}\right\rangle = \left|+_{(\phi_i+\pi)}\right\rangle$.
For qubit i the corrections accumulate as follows:

1. An X-correction from $f^{-1}(i)$ the qubit that its flow is i.
2. A Z-correction from all qubits $j \neq i$ that their flow $f(j)$ is neighbour to i.

$$\boxed{\phi_i' = (-1)^{s_{f^{-1}(i)}} \phi_i + \pi \left(\sum_{j : i \in N_G(f(j)) | j \neq i} s_j \right)}$$

Example



Adapting measurement angles

Example

- Each qubit has result s_i
- The flow is defined as $f(i) = i + 3$
- The order of measurements is the same as the labels
- Consider qubit 5:
  - $f^{-1}(5) = 2$
  - $N_G(5) = \{2, 4, 6, 8\}$
  - X-correction from s_2
  - Z-corrections from: $\{\mathbf{s_1}, \mathbf{s_3}\}$, since we look for $f^{-1}(\cdot)$ for each of the neighbours of 5. Qubit 2 has no past, qubit 8 has our qubit to its past, which leaves only the past of qubit 4 and qubit 6.

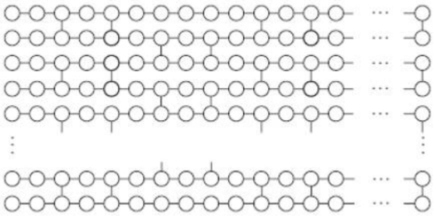
We then obtain the corrected measurement angle:

$$\phi'_5 = (-1)^{s_2}\phi_5 + \pi(s_1 + s_3)$$

Note: Depends on outcomes measured **before** qubit 5.

Brickwork State

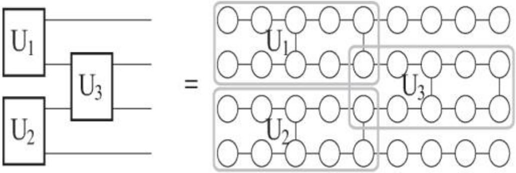
Universal cluster state that does not require Z measurements for breaking the graph. Required for UBQC (see next lecture)



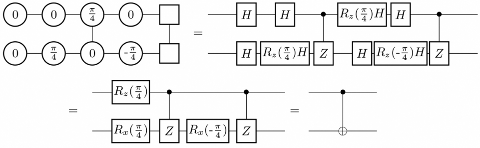
The basis for the global entangled state is the open graph $G_{n,m}$, where $m = 5 \mod 8$. This can be constructed with the following steps:

1. Assign to each vertex an index (i, j), where $1 \leq i \leq n$ is the row and $1 \leq j \leq m$ is the column.
2. For each row i and for all $1 \leq j \leq m-1$ connect vertices (i, j) and $(i, j+1)$ with an edge.
3. For each column $j = (3 \mod 8)$ and each **odd** row i connect vertices (i, j) and $(i+1, j)$ and also vertices $(i, j+2)$ and $(i+1, j+2)$.
4. For each column $j = (7 \mod 8)$ and each **even** row i connect vertices (i, j) and $(i+1, j)$ and also vertices $(i, j+2)$ and $(i+1, j+2)$.

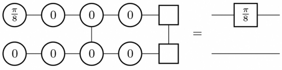
- We can merge different gates at this graph. An example of three unitaries of same size (sufficient for any universal gate) is the following:



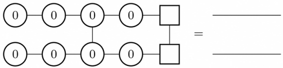
- The order of measurements is column by column
- A universal gate set $\{I, H, \pi/4, \wedge X\}$ can be built from same building block of 10 qubits



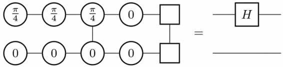
Implementation of a CTRL-X



Implementation of a $\pi/8$ gate



Implementation of the identity



Implementation of a Hadamard gate

Depth and Parallelism

- Storing quantum information (quantum states) without being affected from noise, is a very difficult task.
- It is therefore desirable to try make in parallel as many operations as possible.
- Also theoretically important aspect.
- Partial order of measurements means that any two qubits that are neither at the past nor the future of each other, can be measured in parallel.
- One can construct simplified graph that attempts to make as many measurements in parallel as possible
- Depth of a flow $\Rightarrow$ length of longest chain w.r.t. POSET [Chain: subset S of set C such that S is a total order, i.e. $\forall\ e_i, e_j \in S$ either $e_i \leq e_j$ or $e_j \leq e_i$]
- Finding the flow with the smallest depth for a given computation $\Rightarrow$ maximum parallelise of MBQC pattern.

MBQC Summary

- Start with a universal graph state such as the brickwork state.
- The computation is performed by measuring one-by-one the qubits using single-qubit bases either $\{|+_\theta\rangle, |-_\theta\rangle\}$ or $\{|0\rangle, |1\rangle\}$. Exact basis the qubits are measured depends on previous outcomes.
- The order of measurements is determined by the flow $f(i)$ along with partial order $\preceq$.
- Single qubit gates performed using $J(\theta)$-gate and along with $\wedge Z$ is universal. Random corrections need to be cancelled.
- A unitary U is implemented by a set of default angles ϕ_i if all measurement had outcomes $s_i = 0$.
- The actual basis that a qubit i is measured is modified, using the flow and the stabiliser properties we obtain:

$$\phi_i' = (-1)^{s_{f^{-1}(i)}} \phi_i + \pi \left(\sum_{j : i \in N_G(f(j))} s_j \right)$$

References for MBQC

1. One-way Quantum Computation - a tutorial introduction, D. Browne and H. Briegel, arxiv:quant-ph/0603226
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