Introduction to Quantum Computing
Lecture 7: Complexity and Quantum Algorithms I

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9th October 2018

Notation & Definitions

- Computational Complexity: Classification of problems according to their difficulty. We usually measure the **amount of resources** (e.g . time, space, gates) used by an algorithm as a function of the **input size** n.

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- Notation:

- $f(n)$ is in $O(g(n))$ if for some constant m there exists a positive constant such that $f(n) \leq cg(n)$ for all $n \geq m$
- $f(n)$ is in $\Omega(g(n))$ if for some constant m there exists a positive constant c such that $f(n) \geq cg(n)$ for all $n \geq m$
- $f(n)$ is in $\Theta(g(n))$ if for some constant m there exists positive constants $c_1 \leq c_2$ such that $c_1 g(n) \leq f(n) \leq c_2 g(n)$ for all $n \geq m$

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5. BQP: Solved by probabilistic *quantum computer* in poly time with bounded error (quantum computer solves efficiently)

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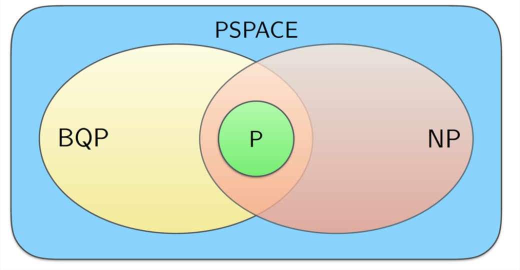
Conjectured Relations: (based on other plausible assumptions)



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Conjectured Relations: (based on other plausible assumptions)

1. There are problems outside NP that quantum computers **can** solve
2. There are problems in NP that quantum computers **cannot** solve (therefore NP-complete problems should be outside BQP)

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(1) Church-Turing Thesis (original)

All physical computable functions can be computed by a **Turing machine**

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- If as conjectured $\mathrm{BPP} \subset \mathrm{BQP}$ then **(2) is wrong**!

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A **quantum** Turing machine can **efficiently** simulate any realistic model of computation

- We are given a classical gate corresponding to an unknown function f as a **black box** (oracle)

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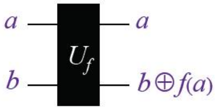

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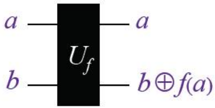

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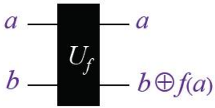

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  By linearity, we can also query in superposition:

$$\sum_{a,b} C_{a,b}\,|a\rangle\,|b\rangle \to \sum_{a,b} C_{a,b}\,|a\rangle\,|b \oplus f(a)\rangle$$

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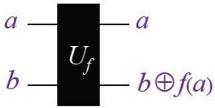

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- Initial protocol by **Deutsch** 1985, improved by **Jozsa**. Current version, is result of further research (**Cleve**, **Ekert**, **Macchiavello** and **Mosca**)

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- **Output:** **Determine** whether the function is **constant** or is **balanced** with the **fewest queries**
- **Performance:**
  1. Classical: To know with certainty we need at least $2^n/2 + 1$ queries
  2. Quantum: With a **single** oracle query

- Recall that U_f is defined as:

$$\sum_{x,y} C_{x,y} |x\rangle |y\rangle \rightarrow \sum_{x,y} C_{x,y} |x\rangle |y \oplus f(x)\rangle$$

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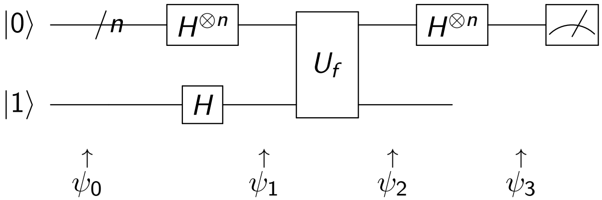

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Property:

$$H|x\rangle = \frac{1}{\sqrt{2}}\sum_{y\in\{0,1\}}(-1)^{xy}|y\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + (-1)^{x}|1\rangle\right)$$

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2. Apply H to all qubits:

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If $f(x)$ is balanced, half terms are $+1$ and half terms are -1 resulting to Eq. (1) giving $p(0) = 0$

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- It constitutes the first **exponential quantum speed-up**. From exponential many oracle calls for P algorithms, we succeed with a single oracle query in EQP!
- It does not give a speed-up compared to BPP, since if we allow for (small) probability of failure, there exist efficient classical algorithms with constant oracle calls (example?)

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2. $[0.x_1 \cdots x_m] = \sum_{k=1}^{m} x_k 2^{-k}$, e.g. $[0.x_1 x_2] = \frac{x_1}{2} + \frac{x_2}{2^2}$

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- **Definition** (classical): The **Discrete Fourier Transform** (DFT) takes a N-dimensional complex vector $(a_0, \cdots, a_{N-1})$ and maps it to $(b_0, \cdots, b_{N-1})$ in this way:

$$b_y = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} a_x \exp(2\pi i x y / N) \qquad (3)$$

Quantum Fourier Transform

- **Definition** (quantum): The **Quantum Fourier Transform** (QFT) is a unitary operation F that performs DFT to the **amplitudes** of a quantum state:

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- To determine a quantum operation it suffices to know how it acts on computational basis state and then extend by linearity
- The QFT acts as (note that $N = 2^n$):

$$F|x_1x_2\cdots x_n\rangle = \frac{1}{\sqrt{N}}\left(|0\rangle + e^{2\pi i[0.x_n]}|1\rangle\right)\otimes\left(|0\rangle + e^{2\pi i[0.x_{n-1}x_n]}|1\rangle\right)\otimes \otimes\cdots\otimes\left(|0\rangle + e^{2\pi i[0.x_1x_2\cdots x_n]}|1\rangle\right) \quad (4)$$

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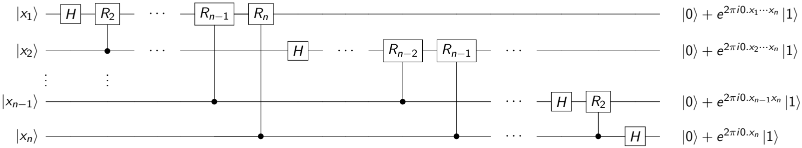

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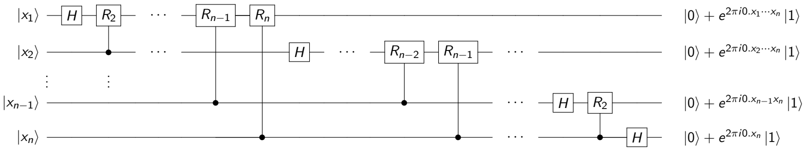

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- It is simple to see that F is unitary since the circuit consists of unitary gates (see also Appendix)

Quantum Fourier Transform

Example: Three qubits

$$F\,|x_1x_2x_3\rangle = |\psi_1\psi_2\psi_3\rangle$$

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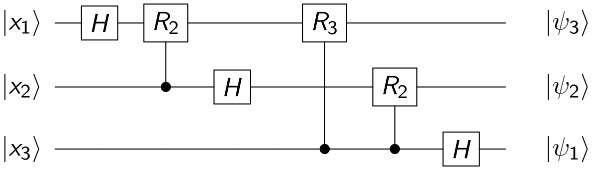

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- To implement the **classical Fast Fourier Transform** $\Theta(n2^n)$ gates are needed

Quantum Fourier Transform

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- To implement the **classical Fast Fourier Transform** $\Theta(n2^n)$ gates are needed
- It appears we obtained an exponential speed-up for a task (DFT) that has many application

Quantum Fourier Transform

- The number of gates in QFT (including the final swaps) is $\Theta(n^2)$
- To implement the **classical Fast Fourier Transform** $\Theta(n2^n)$ gates are needed
- It appears we obtained an exponential speed-up for a task (DFT) that has many application
- However, we **cannot access**(read-out) the amplitudes of a quantum state, so we cannot extract the classical values of the DFT.
- To achieve real speed-up, we need to use QFT as part of larger algorithm (see later!)

Appendix: QFT proof details

- Rewrite Eq. (4):

$$
\begin{aligned}
F\left|x_1 x_2 \cdots x_n\right\rangle &= \left|\psi_1 \psi_2 \cdots \psi_n\right\rangle \\
&= \frac{1}{\sqrt{N}} \left(\sum_{y_1 \in \{0,1\}} e^{2\pi i [0.x_n] y_1} \left|y_1\right\rangle \right) \otimes \left(\sum_{y_2 \in \{0,1\}} e^{2\pi i [0.x_{n-1} x_n] y_2} \left|y_2\right\rangle \right) \otimes \\
&\quad \otimes \cdots \otimes \left(\sum_{y_n \in \{0,1\}} e^{2\pi i [0.x_1 x_2 \cdots x_n] y_n} \left|y_n\right\rangle \right) \\
&= \frac{1}{\sqrt{N}} \left(\sum_{y_1 \in \{0,1\}} e^{2\pi i x (y_1/2^1)} \left|y_1\right\rangle \right) \otimes \left(\sum_{y_2 \in \{0,1\}} e^{2\pi i x (y_2/2^2)} \left|y_2\right\rangle \right) \otimes \\
&\quad \otimes \cdots \otimes \left(\sum_{y_n \in \{0,1\}} e^{2\pi i x (y_n/2^n)} \left|y_n\right\rangle \right) \\
&= \frac{1}{\sqrt{N}} \otimes_{l=1}^{N} \left(\sum_{y_l \in \{0,1\}} e^{2\pi i x (y_l/2^l)} \left|y_l\right\rangle \right) \qquad (5)
\end{aligned}
$$

Appendix: QFT proof details

It follows that

$$\begin{aligned} F\left|x_1x_2\cdots x_n\right\rangle &= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi i x \sum_{l=1}^{n} y_l/2^l}\left|y\right\rangle \\ &= \frac{1}{\sqrt{N}}\sum_{y=0}^{N-1} e^{2\pi i xy/N}\left|y\right\rangle \qquad (6) \end{aligned}$$

where we used that $y = y_1 2^{n-1} + y_2 2^{n-2} + \cdots + y_n 2^0$, similarly for x and $N = 2^n$.

Appendix: QFT proof details

- Express F as an operator: $F = \frac{1}{\sqrt{N}} \sum_{x,y=0}^{N-1} e^{2\pi i xy/N} |y\rangle \langle x|$
- Show that is unitary:

$$
\begin{aligned}
F^\dagger F &= \frac{1}{N} \sum_{x,y,x',y'=0}^{N-1} e^{-2\pi i x'y'/N} |x'\rangle \langle y'| e^{2\pi i xy/N} |y\rangle |x\rangle \\
&= \frac{1}{N} \sum_{x,y,x',y'=0}^{N-1} e^{-2\pi i (x'y'-xy)/N} |x'\rangle \langle x| \langle y'| y\rangle \\
&= \frac{1}{N} \sum_{x,x'=0}^{N-1} \left(\sum_{y=0}^{N-1} e^{-2\pi i y(x'-x)/N} \right) |x'\rangle \langle x| \\
&= \frac{1}{N} \sum_{x,x'=0}^{N-1} N\delta_{x,x'} |x'\rangle \langle x| = \sum_{x=0}^{N-1} |x\rangle \langle x| = \mathbf{1} \ \square
\end{aligned}
$$